THE CHINESE UNIVERSITY OF HONG KONG
MATH3270B
HOMEWORK4 SOLUTION

By Picard's method, we have $\varphi_n(t) = \int_0^t 3t^2(1+\varphi_{n-1}(s))ds$. We choose $\varphi_0 = 0$ and calculate each term.

$$\begin{aligned}\varphi_1 &= \int_0^t 3s^2(1+0)ds = t^3,\\ \varphi_2 &= \int_0^t 3s^2(1+s^3)ds = t^3 + \frac{1}{2}t^6,\\ \varphi_3 &= \int_0^t 3s^2(1+s^3+\frac{1}{2}s^6)ds = t^3 + \frac{1}{2}t^6 + \frac{1}{2\cdot 3}t^9.\end{aligned} \tag{1}$$

From (1), we can assume $\varphi_n (n \geq 1)$ has the form:

$$\varphi_n = \sum_{k=1}^{n} \frac{(t^3)^k}{k!} \tag{2}$$

and then use mathematical induction to prove it. For $n = 1$, (2) is satisfied. Now assume for $n = k$, (2) is satisfied, we must show for $n = k+1$, (2) is also true.

$$\begin{aligned}\varphi_{k+1} &= \int_0^t 3s^2(1 + \sum_{k=1}^{n}\frac{(s^3)^k}{k!})ds\\ &= \int_0^t (3s^2 + \sum_{k=1}^{n}\frac{3s^{3k+2}}{k!})ds\\ &= t^3 + \frac{1}{2}t^6 + \frac{1}{2\cdot 3}t^9 + ... + \frac{t^{3k+3}}{(k+1)!}\\ &= \sum_{k=1}^{n+1}\frac{(t^3)^k}{k!},\end{aligned} \tag{3}$$

which completes our assumption. Next we want to show the convergence of φ_n. By ratio test, we have

$$\left|\frac{\frac{(t^3)^{k+1}}{(k+1)!}}{\frac{(t^3)^k}{k!}}\right| = \frac{t^3}{k+1} \to 0 \tag{4}$$

when $k \to \infty$. Hence, the Picard's sequence converges and the limit function $\varphi(t) = e^{t^3} - 1$, which is the sol to the O.D.E.